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Differential Geometry and its Applications 24 (2006) 613–627

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

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Isometric immersions of $\mathbb{L}^2$ into $\mathbb{L}^4$

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Received 8 June 2005

Available online 4 May 2006

Communicated by O. Kowalski

Abstract

We give a global Weierstrass representation for isometric immersions with flat normal bundle from domains of the Lorentz plane $\mathbb{L}^2$ into $\mathbb{L}^4$. This representation relies on the choice of two holomorphic data on a Riemann surface, and the integration of a hyperbolic linear differential system. As applications, we give classification results and construct complete examples in explicit coordinates by exact integration of the differential system for some particular choices of the holomorphic data.

MSC: 53C40; 53C50

Keywords: Flat surface; Timelike surface; Isometric immersion; Flat normal bundle; Weierstrass representation; de Sitter space

1. Introduction

Just as curvature is the fundamental concept of both Riemannian and semi-Riemannian geometry, the most basic problem in the theory of Riemannian and semi-Riemannian submanifolds is the classification of isometric immersions between model spaces of constant curvature. Generically, the semi-Riemannian theory follows the path suggested by the Riemannian one, but nevertheless there are deep and important differences between both, even in the simplest case of isometric immersions $\mathbb{L}^n \to \mathbb{L}^{n+p}$ between Lorentz spaces.

One of the most significative differences is the existence of a certain type of isometric immersions of $\mathbb{L}^2$ into $\mathbb{L}^3$, discovered by Graves [13] and named by him as *B-scrolls*. These examples and their natural generalizations have non-diagonalizable shape operator and have played a fundamental role in many different problems in Lorentz geometry, from classification of isometric immersions between space forms [6,13,14], to classification of submanifolds by means of characteristic equations [1,15] and results on Lorentzian Willmore submanifolds [2,3].

Another key difference concerns the case of higher codimension. In Euclidean space, the generalized Hartman–Nirenberg theorem ensures that an isometric immersion from $\mathbb{E}^n$ into $\mathbb{E}^{n+p}$ with $p < n$ is a generalized cylinder. In

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0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2006.04.006

addition, there is a vast number of isometric immersions from $\mathbb{E}^2$ into $\mathbb{E}^4$, many of them flat tori in $\mathbb{S}^3 \subset \mathbb{E}^4$ and already described by Bianchi in the 19th century (see also [12]).

In the Lorentzian case, there is a richer variety of isometric immersions from $\mathbb{L}^n$ into $\mathbb{L}^{n+1}$ [13], and a Lorentzian version of the generalized Hartman–Nirenberg theorem is very likely to exist [4]. However, there are stronger restrictions than in the Euclidean case when dealing with isometric immersions of $\mathbb{L}^2$ into $\mathbb{L}^4$, as the following classic result by Dajczer and Nomizu [6] shows: *any isometric immersion of* $\mathbb{L}^2$ *into the de Sitter* 3*-space* $\mathbb{S}^3_1 \subset \mathbb{L}^4$ *takes the form of a hyperbolic cylinder*. Another restriction appearing in this context is the known fact that there are no compact timelike submanifolds in $\mathbb{L}^n$.

In this paper we study isometric immersions of $\mathbb{L}^2$ into $\mathbb{L}^4$ with flat normal bundle, being of course this last hypothesis completely standard in the cases of large codimension, as ours.

Following the explanation in [5] it comes clear that to obtain a general description of such surfaces one needs, even in the local case, constancy in the dimension of the first normal space of the immersion, N_1. We shall restrict ourselves to the situation in which $\dim(N_1) \equiv 2$, which is the most important and studied case.

The basic result of this paper is a Weierstrass representation for Lorentzian flat surfaces immersed into $\mathbb{L}^4$ with flat normal bundle and $\dim N_1 = 2$, via which we describe explicitly such surfaces in terms of holomorphic data. In general, the existence of a holomorphic representation for a class of surfaces is not easily overestimated, since the powerful theorems of complex analysis usually turn into interesting global results for these surfaces.

Contrary to the Riemannian case, the application of methods from complex analysis to Lorentzian surfaces is not common at all. Indeed, even though an oriented Riemannian surface always has associated with its metric a Riemann surface structure, Lorentzian surfaces inherit instead a *Lorentz surface* structure [18], which has nothing to do with complex analysis. The key idea to parametrize Lorentzian flat surfaces in $\mathbb{L}^4$ by means of holomorphic data, inspired in [9] (see also [10] and [11]), is to use the extrinsic geometry of the immersion to give the surface a Riemann surface structure with respect to which its generalized Gauss map is conformal.

We have organized this paper as follows. In Section 2 we study the generalized Gauss map of a Lorentzian flat surface $\psi : \Sigma \to \mathbb{L}^4$ with flat normal bundle, which we view as a map $\mathcal{G} = (\mathcal{G}^+, \mathcal{G}^-) : \Sigma \to \mathbb{C} \cup \{\infty\} \times \mathbb{C} \cup \{\infty\}$. We show that the rank of $d\mathcal{G}$ agrees with the dimension of the first normal space at every point. The fundamental result of this section is Theorem 2.3, in which we show that there is a canonical Riemann surface structure on Σ with respect to which the coordinates $\mathcal{G}^+, \mathcal{G}^-$ of the Gauss map are meromorphic maps. This steps the path for the aimed Weierstrass representation.

In Section 3 we give a Weierstrass representation for all Lorentzian flat surfaces immersed in $\mathbb{L}^4$ with flat normal bundle and regular Gauss map (Theorem 3.1). This result shows how to construct explicitly all such surfaces by means of the choice of two nowhere vanishing holomorphic 1-forms on a Riemann surface, and the integration of a hyperbolic linear differential system. Along with Theorem 3.1 we characterize singular points and uniqueness of the Weierstrass data. Regarding the aforementioned differential system, we study its integrability and give an explicit superposition formula, via which one may obtain new solutions from previously known ones.

Section 4 applies the Weierstrass representation to the obtention of examples. First, we show that every Lorentzian flat surface in $\mathbb{L}^4$ with flat normal bundle and regular Gauss map has at every point exactly two normal fields whose shape operators have rank one. Then we characterize in terms of the Weierstrass data when these normal vector fields are parallel in the normal bundle, and use Theorem 3.1 to present all the resulting surfaces in explicit coordinates. At last, we classify the complete examples of this family, all of which have parabolic conformal structure.

Finally in Section 5 we explore several applications of the representation formula for the case in which the surface lies in a de Sitter 3-space. Particularly, we give two characterization results of the hyperbolic cylinder, including a simple proof of the above quoted Dajczer–Nomizu theorem.

2. The Gauss map of flat timelike surfaces

Let $\mathbb{L}^4$ denote the Lorentz–Minkowski spacetime, that is, the real vector space $\mathbb{R}^4$ with canonical coordinates x_0, x_1, x_2, x_3 and the Lorentzian metric

$$\langle , \rangle = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

We are interested in surfaces immersed in $\mathbb{L}^4$. A smooth immersion $\psi : \Sigma \to \mathbb{L}^4$ of a 2-dimensional manifold Σ into $\mathbb{L}^4$ is called a *timelike surface* if the metric induced on Σ from $\mathbb{L}^4$ via ψ is a Lorentzian metric. We shall also denote this metric by $\langle,\rangle$. If the curvature of the Lorentzian surface $(\Sigma,\langle,\rangle)$ vanishes identically, $K\equiv 0$, the surface will be called *flat*.

The following general result lets us construct on any simply connected Lorentzian flat surface a useful global coordinate immersion.

Lemma 2.1. *Every simply connected Lorentzian flat surface $(\Sigma,\langle,\rangle)$ can be isometrically immersed into the Lorentz plane $\mathbb{L}^2$.*

Proof. Since Σ is flat, there exist $E_1, E_2 \in \mathfrak{X}(\Sigma)$ parallel vector fields on Σ such that $\langle E_1, E_1\rangle = -\langle E_2, E_2\rangle = 1$ and $\langle E_1, E_2\rangle = 0$. Then

$$\langle,\rangle_R = \langle,\rangle + 2\big(E_2^\flat \otimes E_2^\flat\big) \tag{2.1}$$

is a Riemannian metric on Σ. Moreover, the associated connections to $\langle,\rangle$ and $\langle,\rangle_R$ coincide, what shows that $\langle,\rangle_R$ is flat and E_1, E_2 are also parallel for $\langle,\rangle_R$. By Koebe's Uniformization Theorem, there exists a global conformal parameter $w:\Sigma\to\mathbb{C}$ for $\langle,\rangle_R$ such that $\langle,\rangle_R = e^{2\rho}|dw|^2$ for a smooth real function ρ on Σ. Then flatness of $\langle,\rangle_R$ indicates that $\rho_{w\bar{w}}=0$, i.e. ρ is harmonic. Thus, if we take a holomorphic map h on Σ with $\mathrm{Re}(h)=\rho$, and

$$\zeta = x + iy = \int e^h\,dw,$$

we obtain

$$\langle,\rangle_R = |e^h\,dw|^2 = |d\zeta|^2 = dx^2 + dy^2. \tag{2.2}$$

Now, let us consider the parallel vector fields $e_1 = d\zeta(E_1)$, $e_2 = d\zeta(E_2)$, and let $\widetilde{e_1}, \widetilde{e_2}$ denote their corresponding parallel extensions to $\mathbb{R}^2$. Hence there is a unique Lorentzian metric $\langle,\rangle_L$ on $\mathbb{R}^2$ such that

$$\langle\widetilde{e_1},\widetilde{e_1}\rangle_L = -\langle\widetilde{e_2},\widetilde{e_2}\rangle_L = 1, \qquad \langle\widetilde{e_1},\widetilde{e_2}\rangle = 0. \tag{2.3}$$

Then, from (2.1), (2.2) and (2.3) the map $\zeta:(\Sigma,\langle,\rangle)\to(\mathbb{R}^2,\langle,\rangle_L)$ is an isometric immersion. But since $(\mathbb{R}^2,\langle,\rangle_L)$ is globally isometric to $\mathbb{L}^2$, we are done. □

From this lemma, given a timelike flat surface $\psi:\Sigma\to\mathbb{L}^4$, there is a coordinate immersion $x+iy:\Sigma\to\mathbb{C}$ with respect to which the induced metric on Σ is given by $\langle,\rangle = 2\,dx\,dy$.

In addition, let us consider $\{\xi,\hat{\xi}\}$ an orthonormal frame of the (spacelike) normal bundle of ψ, oriented so that $\det(\psi_x,\xi,\hat{\xi},\psi_y)>0$. Then the structure equations of the flat surface $\psi:\Sigma\to\mathbb{L}^4$ read

$$\begin{cases}\psi_{xx} = E_1\xi + E_2\hat{\xi},\\ \psi_{xy} = F_1\xi + F_2\hat{\xi},\\ \psi_{yy} = G_1\xi + G_2\hat{\xi},\end{cases} \qquad \begin{cases}\xi_x = -F_1\psi_x - E_1\psi_y + A\hat{\xi},\\ \xi_y = -G_1\psi_x - F_1\psi_y + B\hat{\xi},\\ \hat{\xi}_x = -F_2\psi_x - E_2\psi_y - A\xi,\\ \hat{\xi}_y = -G_2\psi_x - F_2\psi_y - B\xi,\end{cases}$$

where E_i, F_i, G_i, A, B are smooth real functions on Σ.

From now on, we will assume that the timelike flat surface $\psi:\Sigma\to\mathbb{L}^4$ we are dealing with has flat normal bundle, i.e. its normal curvature tensor vanishes identically, $R^\perp\equiv 0$. From this condition we obtain that

$$0 = R^\perp(\partial_x,\partial_y)\xi = (B_x - A_y)\hat{\xi},$$

what gives $B_x = A_y$ and therefore ensures the existence of a smooth real function μ on Σ such that $d\mu = A\,dx + B\,dy$. This indicates that the orthonormal frame $\{N,\hat{N}\}$ of the normal bundle of ψ given by

$$N = \cos\mu\xi - \sin\mu\hat{\xi}, \qquad \hat{N} = \sin\mu\xi + \cos\mu\hat{\xi}$$

is parallel, i.e. $dN, d\hat{N}\in\mathfrak{X}(\Sigma)$. Moreover, the pair $N,\hat{N}$ is unique up to a constant rotation in the normal bundle of ψ. In this way, the Codazzi–Mainardi equations let us easily obtain

$$(E_1)_y = (F_1)_x, \qquad (F_1)_y = (G_1)_x, \qquad (E_2)_y = (F_2)_x, \qquad (F_2)_y = (G_2)_x,$$

where E_i, F_i, G_i are the corresponding functions associated to $\{N, \hat{N}\}$. Hence, there exist $\varphi, \hat{\varphi} : \Sigma \to \mathbb{R}$ verifying

$$\begin{aligned} &\varphi_{xx} = E_1, \qquad \varphi_{xy} = F_1, \qquad \varphi_{yy} = G_1, \\ &\hat{\varphi}_{xx} = E_2, \qquad \hat{\varphi}_{xy} = F_2, \qquad \hat{\varphi}_{yy} = G_2. \end{aligned}$$

Therefore, the structure equations become

$$\begin{cases} \psi_{xx} = \varphi_{xx} N + \hat{\varphi}_{xx} \hat{N}, \\ \psi_{xy} = \varphi_{xy} N + \hat{\varphi}_{xy} \hat{N}, \\ \psi_{yy} = \varphi_{yy} N + \hat{\varphi}_{yy} \hat{N}, \end{cases} \qquad \begin{cases} -N_x = \varphi_{xy} \psi_x + \varphi_{xx} \psi_y, \\ -N_y = \varphi_{yy} \psi_x + \varphi_{xy} \psi_y, \\ -\hat{N}_x = \hat{\varphi}_{xy} \psi_x + \hat{\varphi}_{xx} \psi_y, \\ -\hat{N}_y = \hat{\varphi}_{yy} \psi_x + \hat{\varphi}_{xy} \psi_y, \end{cases} \tag{2.4}$$

and the integrability conditions of this system turn into the Gauss and Ricci equations, given respectively by

$$\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 = -\left(\hat{\varphi}_{xx}\hat{\varphi}_{yy} - \hat{\varphi}_{xy}^2\right), \qquad \varphi_{xx}\hat{\varphi}_{yy} = \hat{\varphi}_{xx}\varphi_{yy}. \tag{2.5}$$

Next, let us define the *Gauss map* of a timelike flat surface. First, observe that replacing (x, y) by $(-x, -y)$ if necessary, we can suppose that $\psi_x, -\psi_y \in \mathbb{N}^3$, where $\mathbb{N}^3$ is the *positive light cone*

$$\mathbb{N}^3 = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4\colon\ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0,\ x_0 > 0\right\}.$$

The space of null lines of $\mathbb{L}^4$ is the quotient $\mathbb{N}^3/\mathbb{R}^+$, which can be regarded as the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{N}^3$, and is then identified with $\mathbb{C} \cup \{\infty\}$ by

$$\left[(x_0, x_1, x_2, x_3)\right] \in \mathbb{N}^3/\mathbb{R}^+ \longleftrightarrow \frac{x_1 + i x_2}{x_0 - x_3} \in \mathbb{C} \cup \{\infty\} \equiv \mathbb{S}^2_\infty. \tag{2.6}$$

Thus $\mathbb{S}^2_\infty$ has a natural conformal structure. We shall orientate $\mathbb{S}^2_\infty$ as follows: given $p \in \mathbb{N}^3$ we can choose $e_0, e_3 \in \mathbb{L}^4$ with $-\langle e_0, e_0\rangle = 1 = \langle e_3, e_3\rangle$, $\langle e_0, e_3\rangle = 0$, such that $p = e_0 + e_3$. Observe that the first coordinate of e_0 must be positive. Let Π be the timelike plane spanned by e_0, e_3 and choose e_1, e_2 a basis on $\Pi^\perp$. Then e_1, e_2 are induced in the obvious way as tangent vectors to $\mathbb{S}^2_\infty$ at $[p]$. We agree that $\{e_1, e_2\}$ has positive orientation at $T_{[p]}\mathbb{S}^2_\infty$ if $\{e_0 + e_3, e_1, e_2, e_0 - e_3\}$ is a positive basis of $\mathbb{L}^4$. Since this procedure is easily seen to be non-dependent of the choice of e_0, e_3, we get a well defined orientation on $\mathbb{S}^2_\infty$.

With this, the *Gauss map* of $\psi : \Sigma \to \mathbb{L}^4$ is the map

$$\mathcal{G} = (\mathcal{G}^+, \mathcal{G}^-) : \Sigma \to \mathbb{S}^2_\infty \times \mathbb{S}^2_\infty$$

such that $\mathcal{G}^+(p) = [\psi_x(p)]$ and $\mathcal{G}^-(p) = [-\psi_y(p)]$ for all $p \in \Sigma$. Observe that this is nothing but a way to parametrize the usual generalized Gauss map of a timelike surface in $\mathbb{L}^4$, with values in the Grassmannian of oriented timelike planes of $\mathbb{L}^4$.

On the other hand, let $N_1^\psi(p)$ be the *first normal space* of ψ at $p \in \Sigma$ (see [5,7,8] for instance). In our situation, $N_1^\psi(p)$ is the space spanned by the normal vectors $\psi_{xx}(p), \psi_{xy}(p), \psi_{yy}(p)$, and so we have:

Lemma 2.2. *Let $\psi : \Sigma \to \mathbb{L}^4$ be a timelike flat surface with $R^\perp \equiv 0$. Then* $\operatorname{ran}(d\mathcal{G})(p) = \dim(N_1^\psi)(p)$ *for all $p \in \Sigma$.*

Proof. It is obvious that $\dim(N_1^\psi)(p) = 0$ if and only if $\operatorname{ran}(d\mathcal{G})(p) = 0$. On the other hand, if $\operatorname{ran}(d\mathcal{G})(p) = 1$ then $([\psi_x], [\psi_y])_x(p)$, $([\psi_x], [\psi_y])_y(p)$ are linearly dependent and not both zero, so there exist $\lambda, \mu \in \mathbb{R}$ not both zero, such that

$$\mu\left([\psi_x], [\psi_y]\right)_x(p) + \lambda\left([\psi_x], [\psi_y]\right)_y(p) = 0.$$

Thus, if we put $[\psi_x] = a\psi_x$, $[\psi_y] = b\psi_y$ where a, b are nowhere vanishing smooth functions on Σ, we get

$$\mu a(p)\psi_{xx}(p) + \lambda a(p)\psi_{xy}(p) = 0, \qquad \mu b(p)\psi_{xy}(p) + \lambda b(p)\psi_{yy}(p) = 0.$$

Hence $\psi_{xx}(p), \psi_{xy}(p), \psi_{yy}(p)$ are collinear and $\dim(N_1^\psi)(p) = 1$.

Finally, it remains to show that if $\operatorname{ran}(d\mathcal{G})(p) = 2$, then $\dim(N_1^\psi)(p) = 2$. First, note that given a map $\gamma : \Sigma \to \mathbb{N}^3$, if the quotient map $[\gamma] : \Sigma \to \mathbb{N}^3/\mathbb{R}^+$ satisfies that $d[\gamma]_p(v) \neq 0$ for $p \in \Sigma$, $v \in T_p\Sigma$, then $d\gamma_p(v) \neq 0$. Therefore, since $\mathcal{G}$ is an immersion, the map $p \longmapsto (\psi_x(p), \psi_y(p))$ is also an immersion from Σ to $\mathbb{L}^4 \times \mathbb{L}^4$, that is, $(\psi_x, \psi_y)_x(p)$ and $(\psi_x, \psi_y)_y(p)$ are linearly independent. Then, the linear homogeneous system

$$\begin{cases} \lambda\psi_{xx}(p) + \mu\psi_{xy}(p) = 0, \\ \lambda\psi_{xy}(p) + \mu\psi_{yy}(p) = 0 \end{cases}$$

only has the trivial solution. Note that if $\psi_{xx}(p) = 0$, by (2.4), (2.5) we get that $\psi_{xy}(p) = 0$, and the system would have non-trivial solutions, a contradiction. Analogously, we see that $\psi_{yy}(p) \neq 0$. At last, if $\psi_{xy}(p) = 0$, then (2.4), (2.5) provide $\varphi_{yy}^2 = -\hat{\varphi}_{yy}^2$, i.e. $\psi_{yy}(p) = 0$, which is impossible.

Therefore, $\psi_{xx}, \psi_{xy}, \psi_{yy}$ do not vanish at p and, since the above system only admits the trivial solution, they are not collinear. In other words, $\dim(N_1^{\psi})(p) = 2$. $\square$

We shall deal with the case in where $\mathcal{G}$ is regular, which is by far the most interesting situation. Indeed, the following result exposes the geometric importance of timelike flat surfaces with flat normal bundle and regular Gauss map, and plays a key role in the description of such surfaces through a Weierstrass-type representation (see next section).

Theorem 2.3. *Let $\psi : \Sigma \to \mathbb{L}^4$ be a timelike flat surface with flat normal bundle and regular Gauss map. There exists a canonical Riemann surface structure on Σ with respect to which $\mathcal{G}^+, \mathcal{G}^- : \Sigma \to \mathbb{C} \cup \{\infty\}$ are meromorphic maps, and $\mathcal{G} : \Sigma \to \mathbb{S}^2_\infty \times \mathbb{S}^2_\infty$ is conformal.*

Proof. As we saw in the proof of Lemma 2.2, none of the vectors $\psi_{xx}, \psi_{xy}, \psi_{yy}$ vanish at any point, since $\mathcal{G}$ is regular.

Let us define $\zeta = s + it = \varphi_x + i\hat{\varphi}_x$. Since ψ_{xx}, ψ_{xy} are linearly independent, $\zeta : \Sigma \to \mathbb{C}$ is a global coordinate immersion, and endows Σ with a Riemann surface structure.

In the same way, $w = u + iv = \hat{\varphi}_y + i\varphi_y : \Sigma \to \mathbb{C}$ is also a global coordinate immersion. Moreover, from (2.4) we obtain the following Cauchy–Riemann equations on Σ:

$$\frac{\partial u}{\partial s} = \frac{\hat{\varphi}_{xy}^2 - \hat{\varphi}_{xx}\hat{\varphi}_{yy}}{\varphi_{xx}\hat{\varphi}_{xy} - \varphi_{xy}\hat{\varphi}_{xx}} = \frac{\partial v}{\partial t}, \qquad -\frac{\partial u}{\partial t} = \frac{\varphi_{xx}\hat{\varphi}_{yy} - \varphi_{xy}\hat{\varphi}_{xy}}{\varphi_{xx}\hat{\varphi}_{xy} - \varphi_{xy}\hat{\varphi}_{xx}} = \frac{\partial v}{\partial s}.$$

Hence, the Riemann surface structures induced by ζ and w coincide, what ensures the existence of a nowhere vanishing holomorphic function f on Σ such that $dw = f\,d\zeta$. Once here, the definition of ζ and w together with the structure equations (2.4) give

$$\begin{aligned} &2(\psi_x)_\zeta = N - i\hat{N}, \qquad 2(\psi_x)_w = \frac{1}{f}(N - i\hat{N}), \\ &2(\psi_y)_w = \hat{N} - iN, \qquad 2(\psi_y)_\zeta = f(\hat{N} - iN). \end{aligned} \tag{2.7}$$

This shows that $\mathcal{G}^+, \mathcal{G}^- : \Sigma \to \mathbb{S}^2_\infty$, and therefore $\mathcal{G} = (\mathcal{G}^+, \mathcal{G}^-)$ are conformal maps. At last, recalling the way we oriented $\mathbb{S}^2_\infty$ and the condition $\det(\psi_x, N, \hat{N}, \psi_y) > 0$ we infer that both $\mathcal{G}^+, \mathcal{G}^-$ preserve orientation. Thus $\mathcal{G}^+, \mathcal{G}^- : \Sigma \to \mathbb{C} \cup \{\infty\}$ are meromorphic maps, and we are done. $\square$

3. The Weierstrass representation

When dealing with surfaces in $\mathbb{L}^4$, a standard way to simplify computations is to consider $\mathbb{L}^4$ in its Hermitian model. Specifically, let us identify $\mathbb{L}^4$ with the space of 2×2 Hermitian matrices, Herm(2), as

$$(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \mathrm{Herm}(2).$$

The metric $\langle , \rangle$ on this model is

$$\left\langle \begin{pmatrix} a_1 & b_1 \\ \overline{b_1} & c_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ \overline{b_2} & c_2 \end{pmatrix} \right\rangle = -\frac{1}{2}\{a_1c_2 + a_2c_1 - b_1\overline{b_2} - b_2\overline{b_1}\},$$

and consequently $\langle m, m\rangle = -\det(m)$ for all $m \in \mathrm{Herm}(2)$. In addition, the complex Lie group $\mathbf{SL}(2,\mathbb{C})$ acts naturally on $\mathbb{L}^4$ through the action

$$\Phi \in \mathbf{SL}(2,\mathbb{C}) \mapsto \Phi \cdot m = \Phi m \Phi^*, \quad m \in \mathrm{Herm}(2), \quad \Phi^* = \bar{\Phi}^t.$$

In this way, $\mathbf{SL}(2,\mathbb{C})$ is identified with the isometry subgroup of $\mathbb{L}^4$ that preserves metric, orientation and time-orientation. The positive null cone $\mathbb{N}^3$ is regarded as the space

$$\mathbb{N}^3 = \left\{ w\overline{w}^t\colon\ w^t = (w_1, w_2) \in \mathbb{C}^2 \setminus (0,0) \right\} \subset \mathrm{Herm}(2),$$

and the *de Sitter space* $\mathbb{S}^3_1 = \{x \in \mathbb{L}^4\colon\ \langle x, x\rangle = 1\}$ is

$$\mathbb{S}^3_1 = \left\{ \Phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Phi^*\colon\ \Phi \in \mathbf{SL}(2,\mathbb{C}) \right\}.$$

Here the vector $w \in \mathbb{C}^2$ is defined up to multiplication by $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Finally, the projection $\mathbb{N}^3 \to \mathbb{N}^3/\mathbb{R}^+ \equiv \mathbb{C} \cup \{\infty\}$ becomes $w\overline{w}^t \to w_1/w_2$, for $w = (w_1, w_2)$.

With all of this, the holomorphic representation for the class of timelike surfaces with $R^\perp \equiv 0$ and regular Gauss map in Herm(2) is the following.

Theorem 3.1 *(Weierstrass representation). Let Σ be a Riemann surface, and consider two nowhere vanishing holomorphic* 1*-forms ϑ_1, ϑ_2 on Σ. If $F : \Sigma \to \mathbf{SL}(2,\mathbb{C})$ is a holomorphic curve satisfying*

$$F^{-1}\, dF = \begin{pmatrix} 0 & \vartheta_1 \\ \vartheta_2 & 0 \end{pmatrix} \tag{3.1}$$

and $\phi : \Sigma \to \mathbb{C}$ verifies the linear differential system

$$\begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix}_{\bar{z}} d\bar{z} = \begin{pmatrix} 0 & \bar{\vartheta}_1/\vartheta_1 \\ \bar{\vartheta}_2/\vartheta_2 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix}_z dz \tag{3.2}$$

on Σ (z being an arbitrary complex coordinate), then the map

$$\psi = F \begin{pmatrix} -\phi_z\, dz/\vartheta_2 & \bar{\phi} \\ \phi & -\bar{\phi}_z\, dz/\vartheta_1 \end{pmatrix} F^* : \Sigma \to \mathrm{Herm}(2) \equiv \mathbb{L}^4 \tag{3.3}$$

defines at its regular points a timelike flat surface in $\mathbb{L}^4$ with flat normal bundle and regular Gauss map, for which $\mathcal{G}^+, \mathcal{G}^- : \Sigma \to \mathbb{C} \cup \{\infty\}$ are $\mathcal{G}^+ = F_{11}/F_{21}$, $\mathcal{G}^- = F_{12}/F_{22}$ (here F_{ij} are the coordinates of $F \in \mathbf{SL}(2,\mathbb{C})$).

Conversely, every simply connected timelike flat surface in $\mathbb{L}^4$ with flat normal bundle and regular Gauss map can be represented in this way with respect to the canonical Riemann surface structure specified in Theorem 2.3.

Proof. We start with the converse. Let $\psi : \Sigma \to \mathbb{L}^4$ be an immersion verifying the conditions of the theorem. Since the Riemann surface Σ is simply connected, we may take a global conformal parameter z on Σ. Let h be the non-vanishing holomorphic function on Σ such that $d\zeta = 2h\, dz$, where here ζ is the holomorphic coordinate immersion defined in Theorem 2.3. Then, from (2.7) we get

$$(\psi_x)_z = h(N - i\hat{N}), \qquad (\psi_y)_z = fh(\hat{N} - iN), \tag{3.4}$$

and from here and the structure equations (2.4),

$$(\psi_x)_{z\bar{z}} = -2|h|^2 \psi_y, \quad (\psi_y)_{z\bar{z}} = -2|fh|^2 \psi_x. \tag{3.5}$$

Besides, by Theorem 2.3, $\mathcal{G}^+ = [\psi_x] : \Sigma \to \mathbb{C} \cup \{\infty\}$ is a meromorphic map, what ensures the existence of two holomorphic functions $A, B : \Sigma \to \mathbb{C}$ and a positive smooth function $\lambda : \Sigma \to \mathbb{R}^+$ such that

$$\psi_x = \lambda \begin{pmatrix} A\bar{A} & A\bar{B} \\ \bar{A}B & B\bar{B} \end{pmatrix}. \tag{3.6}$$

Now, from (3.4) we see that $\langle(\psi_x)_z,(\psi_x)_{\bar z}\rangle = 2|h|^2$, whereas differentiation of (3.6) and the expression of the metric in Herm(2) yield

$$\bigl\langle(\psi_x)_z,(\psi_x)_{\bar z}\bigr\rangle = \frac{\lambda^2}{2}|AB_z - BA_z|^2.$$

Thus $|d\zeta|^2 = \lambda^2|A\,dB - B\,dA|^2$, what implies that $\Delta\log\lambda = 0$ on Σ. Therefore there exists a nowhere vanishing holomorphic function q on Σ with $\lambda = |q|^2$, and this map q can be chosen so that $d\zeta = q^2(A\,dB - B\,dA)$ holds on Σ.

Let us define next the meromorphic curve

$$F = \begin{pmatrix} qA & d(qA)/d\zeta \\ qB & d(qB)/d\zeta \end{pmatrix}.$$

It is immediate that F takes its values in $\mathbf{SL}(2,\mathbb{C})$, and that

$$F^{-1}dF = \begin{pmatrix} 0 & \vartheta_1 \\ \vartheta_2 & 0 \end{pmatrix}, \tag{3.7}$$

where ϑ_1 is a holomorphic 1-form on Σ, and $\vartheta_2 = d\zeta$. Since $|q|^2 = \lambda$, it is clear that

$$\psi_x = F\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}F^*. \tag{3.8}$$

Now, from (3.8) and making use of (3.7), it is easy to obtain from the identities in (3.4) and (3.5) that $\vartheta_1 = ifh\,dz$, and

$$\psi_y = F\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}F^*, \qquad N = F\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}F^*, \qquad \hat N = F\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}F^*. \tag{3.9}$$

Observe besides that (3.4) and (3.5) yield

$$\psi_x = \frac{-1}{2\overline{fh}}(\hat N - iN)_{\bar z}, \qquad \psi_y = \frac{-1}{2\bar h}(N - i\hat N)_{\bar z}. \tag{3.10}$$

Let $a, b, \alpha, \beta : \Sigma \to \mathbb{R}$ be the smooth functions that make the identity

$$\psi = a\psi_x + b\psi_y + \alpha N + \beta\hat N$$

hold. Since $\alpha_z = \langle\psi, N_z\rangle$, $\beta_z = \langle\psi, \hat N_z\rangle$, the expressions in (3.10) give

$$\psi = \frac{-(\alpha+i\beta)_z}{2h}\psi_x - \frac{(\beta+i\alpha)_z}{2fh}\psi_y + \alpha N + \beta\hat N, \tag{3.11}$$

and thus, in particular,

$$\frac{(\alpha+i\beta)_z}{2h} \in \mathbb{R}, \qquad \frac{(\beta+i\alpha)_z}{2fh} \in \mathbb{R}. \tag{3.12}$$

If we denote now $\phi = \alpha + i\beta$, then (3.11) and (3.12) are written respectively as

$$\psi = -\frac{\phi_z}{2h}\psi_x - \frac{i\bar\phi_z}{2fh}\psi_y + \mathrm{Re}(\phi)N + \mathrm{Im}(\phi)\hat N, \tag{3.13}$$

and

$$\begin{pmatrix} \phi \\ \bar\phi \end{pmatrix}_{\bar z} = \frac{\bar h}{h}\begin{pmatrix} 0 & -\bar f/f \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \bar\phi \end{pmatrix}_z. \tag{3.14}$$

Finally, from (3.8), (3.9) and $\vartheta_1 = ifh\,dz$, we see that (3.13) and (3.14) turn into (3.3) and (3.2), respectively. The identities $\mathcal{G}^+ = F_{11}/F_{21}$, $\mathcal{G}^- = F_{12}/F_{22}$ are straightforward. This finishes the proof of the converse.

The direct implication is just a matter of computation. We outline the proof. If $\vartheta_1, \vartheta_2, F, \phi$ satisfy the conditions of the theorem, z is a local complex coordinate of Σ and f, h are holomorphic functions satisfying $\vartheta_2 = 2h\,dz$,

$\vartheta_1 = i f h\, dz$, then (3.3) splits into

$$\psi = -\frac{\phi_z}{2h} E_1 - \frac{i\bar{\phi}_z}{2fh} E_2 + \operatorname{Re}(\phi) N + \operatorname{Im}(\phi)\hat{N},$$

where $N, \hat{N}$ are as in (3.9) and

$$E_1 = F \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F^*, \qquad E_2 = F \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} F^*.$$

A direct computation tells that $N, \hat{N}$ are normal to ψ. In particular, ψ is a timelike surface at its regular points and, since $N, \hat{N}$ are parallel, it has flat normal bundle. Flatness of the metric follows from the fact that $(E_1 + E_2)/\sqrt{2}$ and $(E_1 - E_2)/\sqrt{2}$ conform a parallel orthonormal frame of the tangent bundle of ψ. Finally, the expression of $\mathcal{G}^+, \mathcal{G}^-$ is immediate, while regularity of $\mathcal{G}$ holds since ϑ_1, ϑ_2 do not vanish. □

Concerning Theorem 3.1 there are two basic facts that we must clarify: the appearance of singular points and the uniqueness of the data $\vartheta_1, \vartheta_2, \phi$.

Let $\psi : \Sigma \to \mathbb{L}^4$ be an immersion in the conditions of Theorem 3.1, and choose (x, y) global null coordinates on Σ as well as a global conformal parameter z. Then ψ is expressed as (3.13) in terms of f, h, ϕ and z. Note that $x = x(z)$, $y = y(z)$. Since $\langle \psi_x, \psi \rangle = -i\bar{\phi}_z/(2fh)$, derivation with respect to $\bar{z}$ yields

$$y_{\bar{z}} = -\bar{h}\phi - i\frac{\bar{\phi}_{z\bar{z}}}{2fh}. \tag{3.15}$$

Analogously we obtain

$$x_z = i f h \phi - \frac{\bar{\phi}_{z\bar{z}}}{2\bar{h}}. \tag{3.16}$$

The condition that detects the singular points of ψ is $x_z y_{\bar{z}} - x_{\bar{z}} y_z = 0$, that is, $\operatorname{Im}(x_z y_{\bar{z}}) = 0$. Hence, by (3.15), (3.16), we obtain that the singular points are characterized in terms of f, h, ϕ and z by the condition

$$\left(i f h \phi - \frac{\bar{\phi}_{z\bar{z}}}{2\bar{h}}\right)\left(-\bar{h}\phi - i\frac{\bar{\phi}_{z\bar{z}}}{2fh}\right) \in \mathbb{R}. \tag{3.17}$$

Regarding uniqueness of the data we first observe that, if (x, y) are null coordinates of $\psi : \Sigma \to \mathbb{L}^4$, $(N, \hat{N})$ is a pair of parallel normal sections and z is a global conformal coordinate of Σ, then (3.4), (3.13) ensure that f, h, ϕ are unique. Therefore $\vartheta_2 = 2h\, dz$ and $\vartheta_1 = i f \vartheta_2/2$ are also unique. But now we note that (x, y) and $N, \hat{N}$ are defined up to the changes $(x, y) \mapsto (e^{\lambda} x, e^{-\lambda} y)$ and $N + i\hat{N} \mapsto e^{i\theta}(N + i\hat{N})$ for $\lambda, \theta \in \mathbb{R}$ (recall that $\psi_x \in \mathbb{N}^3$). Because of this, and using (3.4), (3.13), the functions f, h, ϕ are unique up to the transformations

$$h \mapsto e^{\lambda + i\theta} h, \quad f \mapsto e^{-2(\lambda + i\theta)} f, \quad \phi \mapsto e^{i\theta}\phi.$$

In other words, two triplets of data $(\vartheta_1, \vartheta_2, \phi)$, $(\widetilde{\vartheta}_1, \widetilde{\vartheta}_2, \widetilde{\phi})$ determine the same immersion via Theorem 3.1 if and only if they only differ by the change

$$\widetilde{\vartheta}_1 = e^{-(\lambda + i\theta)}\vartheta_1, \quad \widetilde{\vartheta}_2 = e^{\lambda + i\theta}\vartheta_2, \quad \widetilde{\phi} = e^{i\theta}\phi, \quad \lambda, \theta \in \mathbb{R}. \tag{3.18}$$

In the remaining of this section, we study the two differential equations appearing in the representation theorem, namely, (3.1) and (3.2).

The complex differential equation (3.1) was studied in [9]. The following facts are recovered from that work, or are easily obtained from them.

- If ϑ_1, ϑ_2 are nowhere vanishing holomorphic 1-forms on the Riemann surface Σ, and if Σ is simply connected, then (3.1) is integrable. That is, there is a holomorphic curve $F : \Sigma \to \mathbf{SL}(2, \mathbb{C})$ which verifies (3.1). Moreover, a holomorphic curve F in $\mathbf{SL}(2, \mathbb{C})$ solves (3.1) for some ϑ_1, ϑ_2 if and only if its coordinates F_{ij} satisfy $F_{22}\, dF_{11} - F_{12}\, dF_{21} = 0$.

- Let C, D be holomorphic functions on Σ such that $C\,dD - D\,dC = \vartheta_2$. Assume also that, if z is a complex coordinate on Σ with respect to which $\vartheta_2 = 2h\,dz$, $\vartheta_1 = ifh\,dz$, both C, D are linearly independent solutions of the linear differential equation

$$Z'' - \frac{h'}{h}Z' - 2ifh^2 Z = 0, \qquad \left({}' = \frac{d}{dz}\right). \tag{3.19}$$

Then the holomorphic curve $F : \Sigma \to \mathbf{SL}(2, \mathbb{C})$ given by

$$F = \begin{pmatrix} C & dC/\vartheta_2 \\ D & dD/\vartheta_2 \end{pmatrix} \tag{3.20}$$

is a solution to (3.1). This solution is unique up to left multiplication by a constant element of $\mathbf{SL}(2, \mathbb{C})$ [9]. Moreover, different solutions to (3.1) generate via Theorem 3.1 congruent surfaces in $\mathbb{L}^4$.
- From (3.19) and (3.20), the following identities are easily obtained:

$$d(dC/\vartheta_2) = C\vartheta_1, \qquad d(dD/\vartheta_2) = D\vartheta_1.$$

These comments simplify considerably the integration of (3.1).

Next, we focus on the linear differential system (3.2).

First of all, we observe that any $c \in \mathbb{C}$ is a solution to (3.2). If $\phi = c \in \mathbb{C} \setminus \{0\}$, the resulting surface lies in the quadric of $\mathbb{L}^4$ given by $\langle x, x\rangle = |c|^2$, which is a 3-dimensional de Sitter space of constant curvature $1/|c|^2$. This type of surfaces will be studied in Section 5.

On the other hand, the system (3.2) admits an explicit superposition formula. To see this, we first note that (3.2) can be written alternatively as

$$\left[\begin{pmatrix} 0 & h \\ fh & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \bar\phi \end{pmatrix}\right]_{\bar z} = \left[\begin{pmatrix} \bar h & 0 \\ 0 & -\overline{fh} \end{pmatrix}\begin{pmatrix} \phi \\ \bar\phi \end{pmatrix}\right]_z.$$

Thus, there exists a smooth function $\mu = (\mu_1, \mu_2)^T$ such that

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}_z = \begin{pmatrix} 0 & h \\ fh & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \bar\phi \end{pmatrix}, \qquad \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}_{\bar z} = \begin{pmatrix} \bar h & 0 \\ 0 & -\overline{fh} \end{pmatrix}\begin{pmatrix} \phi \\ \bar\phi \end{pmatrix}. \tag{3.21}$$

In particular, $\mu_1 \in \mathbb{R}$ and $\mu_2 \in i\mathbb{R}$ everywhere. Moreover, we also see that $(-fh\mu_1)_z = (h\mu_2)_{\bar z}$, and so there is a complex-valued smooth function χ which satisfies $\chi_z = h\mu_2$ and $\chi_{\bar z} = -\overline{fh}\mu_1$. From this relation and (3.21) we finally see that $\hat\phi = i\chi$ is a new solution of the differential system (3.2). Furthermore, the solution ϕ we started with is written in terms of $\hat\phi$ as

$$\bar\phi = \frac{i\hat\phi_{z\bar z}}{\bar f|h|^2}.$$

Conversely, a direct computation shows that if ϕ is a solution to (3.2), then

$$\hat\phi = \frac{ic\overline{\phi}_{z\bar z}}{f|h|^2}, \quad c \in \mathbb{R}\setminus\{0\}$$

is a new solution to (3.2).

This provides a way to obtain particular explicit integrations of the system (3.2). For instance, if we start with the trivial solution $\phi = 0$ and follow the above process, we end up with the new solution

$$\hat\phi(z, \bar z) = c_1 \int h(z)\,dz + c_2 \overline{\int f(z)h(z)\,dz}, \quad c_1, c_2 \in \mathbb{R}.$$

To conclude this section, we give an alternative formulation of the Weierstrass representation. This description simplifies the differential system, but needs the choice of a particular holomorphic coordinate immersion on Σ rather than an arbitrary coordinate.

Theorem 3.2. *Let Σ be a Riemann surface endowed with a global holomorphic coordinate immersion $\zeta = s + it : \Sigma \to \mathbb{C}$, and let f be a nowhere vanishing holomorphic function on Σ. If $g : \Sigma \to \mathbb{R}$ is a solution of the second order hyperbolic linear differential system*

$$\mathrm{Re}(f)(g_{ss} - g_{tt}) = 2\,\mathrm{Im}(f) g_{st}, \tag{3.22}$$

and $F : \Sigma \to \mathbf{SL}(2, \mathbb{C})$ is a holomorphic curve verifying

$$F^{-1}F_\zeta = \begin{pmatrix} 0 & if/2 \\ 1 & 0 \end{pmatrix}, \tag{3.23}$$

then the map $\psi : \Sigma \to \mathrm{Herm}(2) \equiv \mathbb{L}^4$ given by

$$\psi = F \begin{pmatrix} -g_{\zeta\bar\zeta} & g_\zeta \\ g_{\bar\zeta} & 2ig_{\zeta\zeta}/f \end{pmatrix} F^* \tag{3.24}$$

defines at its regular points a timelike flat surface in $\mathbb{L}^4$ with flat normal bundle and regular Gauss map, for which $\mathcal{G}^+, \mathcal{G}^- : \Sigma \to \mathbb{C} \cup \{\infty\}$ are $\mathcal{G}^+ = F_{11}/F_{21}$ and $\mathcal{G}^- = F_{12}/F_{22}$.

Conversely, if $\psi : \Sigma \to \mathbb{L}^4$ is a simply connected timelike flat surface in $\mathbb{L}^4$ with flat normal bundle and regular Gauss map constructed via Theorem 3.1, and we consider the coordinate immersion

$$\zeta = \int \vartheta_2 : \Sigma \to \mathbb{C}, \tag{3.25}$$

then the surface ψ can be represented following the above procedure.

Proof. For the converse, if we let the coordinate ζ given by (3.25) be the local complex coordinate z appearing in the proof of Theorem 3.1, then the computations developed there hold for the choice $h = 1/2$. In particular the conditions (3.12) provide $\phi_\zeta \in \mathbb{R}$ and $i\bar\phi_\zeta/f \in \mathbb{R}$. From $\phi_\zeta \in \mathbb{R}$ and the simple connectivity of Σ we obtain the existence of a real function $g : \Sigma \to \mathbb{R}$ with $g_{\bar\zeta} = \phi$ (and $g_\zeta = \bar\phi$). Hence $i\bar\phi_\zeta/f \in \mathbb{R}$ gives $g_{\zeta\zeta}/f \in i\mathbb{R}$, that is, g verifies the differential equation (3.22) for $\zeta = s + it$. The rest of the proof is straightforward from Theorem 3.1 and the condition $\phi = g_{\bar\zeta}$. $\square$

4. Examples

In this section we shall describe in explicit coordinates a large family of timelike flat surfaces in $\mathbb{L}^4$ with flat normal bundle and regular Gauss map. To define geometrically this family, we will first show the existence for any such surface of two unique unit normal vector fields with rank one shape operators.

Let $\psi : \Sigma \to \mathbb{L}^4$ be a surface in the above conditions, and choose $\{N, \hat N\}$ an orthonormal basis of parallel vector fields in the normal bundle $\mathfrak{X}^\perp(\Sigma)$. Let $z = s + it$ be a conformal parameter on Σ with respect to its canonical Riemann surface structure defined in Theorem 2.3. Then (3.10) indicates that

$$\begin{pmatrix} N_z \\ \hat N_z \end{pmatrix} = \begin{pmatrix} ihf & -h \\ -fh & ih \end{pmatrix} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}.$$

Any other positive orthonormal basis $\{\mathbf{e}_3, \mathbf{e}_4\}$ of $\mathfrak{X}^\perp(\Sigma)$ is written as $\mathbf{e}_3 + i\mathbf{e}_4 = e^{-i\nu}(N + i\hat N)$ for some smooth function $\nu : \Sigma \to \mathbb{R}$. Hence, the tangent part of the derivatives of $\mathbf{e}_3, \mathbf{e}_4$ are

$$\begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}_z^T = \begin{pmatrix} \cos\nu & \sin\nu \\ -\sin\nu & \cos\nu \end{pmatrix} \begin{pmatrix} ihf & -h \\ -fh & ih \end{pmatrix} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}. \tag{4.1}$$

Now, let $A_{\mathbf{e}_3}, A_{\mathbf{e}_4}$ denote the shape operators associated to the unit normals $\mathbf{e}_3, \mathbf{e}_4$. From the above equation, after a simple computation we obtain that, at a particular point, $A_{\mathbf{e}_3}$ has rank one if and only if $A_{\mathbf{e}_4}$ has rank one, if and only if $\mathrm{Re}(e^{2i\nu} f) = 0$. That is, we obtain

$$\mathbf{e}_3 + i\mathbf{e}_4 = \frac{\pm(1 \pm i)}{\sqrt{2}} \frac{\sqrt{f}}{|\sqrt{f}|}(N + i\hat N) \tag{4.2}$$

for a holomorphic square root $\sqrt{f}$ of f. Since a different choice of parallel normals $N, \hat{N}$ also changes f so that the right hand side of (4.2) remains, the pair $\mathbf{e}_3, \mathbf{e}_4$ is unique up to $\pi/2$-rotations in the oriented normal plane. We remark that $\mathbf{e}_3, \mathbf{e}_4$ *are parallel if and only if f is constant*.

In addition, from (4.1), (4.2), we get the formulas

$$\frac{1}{\sqrt{2}}\begin{pmatrix} (\mathbf{e}_3)_s^T \\ -(\mathbf{e}_3)_t^T \end{pmatrix} = \begin{pmatrix} -|\sqrt{f}|\operatorname{Im}((1+i)\sqrt{f}h) & -\frac{1}{|\sqrt{f}|}\operatorname{Im}((1+i)\sqrt{f}h) \\ |\sqrt{f}|\operatorname{Re}((1+i)\sqrt{f}h) & \frac{1}{|\sqrt{f}|}\operatorname{Re}((1+i)\sqrt{f}h) \end{pmatrix}\begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

and

$$\frac{1}{\sqrt{2}}\begin{pmatrix} (\mathbf{e}_4)_s^T \\ -(\mathbf{e}_4)_t^T \end{pmatrix} = \begin{pmatrix} -|\sqrt{f}|\operatorname{Re}((1+i)\sqrt{f}h) & \frac{1}{|\sqrt{f}|}\operatorname{Re}((1+i)\sqrt{f}h) \\ -|\sqrt{f}|\operatorname{Im}((1+i)\sqrt{f}h) & \frac{1}{|\sqrt{f}|}\operatorname{Im}((1+i)\sqrt{f}h) \end{pmatrix}\begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}.$$

From here it is immediate to check that the unit tangent vector fields $\mathbf{e}_1, \mathbf{e}_2$ given by

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}\left(|\sqrt{f}|\psi_x + \frac{1}{|\sqrt{f}|}\psi_y\right), \qquad \mathbf{e}_2 = \frac{1}{\sqrt{2}}\left(-|\sqrt{f}|\psi_x + \frac{1}{|\sqrt{f}|}\psi_y\right)$$

satisfy that $\mathbf{e}_1 \in \operatorname{Ker}(A_{\mathbf{e}_3})$ and $\mathbf{e}_2 \in \operatorname{Ker}(A_{\mathbf{e}_4})$. Now, $\mathbf{e}_1, \mathbf{e}_2$ are unique up to sign, and *they are parallel in the tangent bundle $\mathfrak{X}(\Sigma)$ if and only if f is constant*.

We shall call $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ the *holonomic frame* of the timelike flat surface ψ.

Next, we investigate when the shape operators associated to $\mathbf{e}_3, \mathbf{e}_4$ are diagonalizable. First, note that if z is an arbitrary complex coordinate, then

$$(\mathbf{e}_3)_z^T = \alpha_1\psi_x + \beta_1\psi_y, \qquad (\mathbf{e}_4)_z^T = \alpha_2\psi_x + \beta_2\psi_y$$

for

$$\begin{cases} \alpha_1 = \frac{-1+i}{\sqrt{2}}|\sqrt{f}|h\sqrt{f}, & \alpha_2 = -\frac{1+i}{\sqrt{2}}|\sqrt{f}|h\sqrt{f}, \\ \beta_1 = \frac{-1+i}{\sqrt{2}}\frac{1}{|\sqrt{f}|}h\sqrt{f}, & \beta_2 = \frac{1+i}{\sqrt{2}}\frac{1}{|\sqrt{f}|}h\sqrt{f}. \end{cases}$$

Therefore, $A_{\mathbf{e}_3}$ (resp. $A_{\mathbf{e}_4}$) is non-diagonalizable at a point if and only if $\operatorname{tr}(A_{\mathbf{e}_3}) = 0$ (resp. $\operatorname{tr}(A_{\mathbf{e}_4}) = 0$), if and only if $\operatorname{Re}(\alpha_1 z_x + \beta_1 z_y) = 0$ (resp. $\operatorname{Re}(\alpha_2 z_x + \beta_2 z_y) = 0$).

Next, observe that

$$-\langle \mathbf{e}_3, \psi_{z\bar{z}}\rangle = (\alpha_1 z_x + \beta_1 z_y)\langle \psi_z, \psi_{\bar{z}}\rangle,$$

what shows that $\alpha_1 z_x + \beta_1 z_y \in \mathbb{R}$. Analogously, $\alpha_2 z_x + \beta_2 z_y \in \mathbb{R}$. Putting together all these facts, and noting that $x_z y_{\bar{z}} - x_{\bar{z}} y_z \neq 0$, we obtain that $A_{\mathbf{e}_3}$ (resp. $A_{\mathbf{e}_4}$) is non-diagonalizable at some point if and only if

$$\alpha_1 y_{\bar{z}} - \beta_1 x_{\bar{z}} = 0 \quad (\text{resp. } \alpha_2 y_{\bar{z}} - \beta_2 x_{\bar{z}} = 0)$$

at that point. In other words, $A_{\mathbf{e}_3}$ (resp. $A_{\mathbf{e}_4}$) is non-diagonalizable exactly at the points described in terms of the data (f, h, ϕ) by the condition

$$|f|y_{\bar{z}} = x_{\bar{z}}, \quad (\text{resp. } |f|y_{\bar{z}} = -x_{\bar{z}}),$$

being $x_{\bar{z}}, y_{\bar{z}}$ given by (3.16), (3.15). In particular, the two shape operators cannot be simultaneously non-diagonalizable at any point.

It comes clear that the normal vectors $\mathbf{e}_3, \mathbf{e}_4$ may have non-diagonalizable shape operator, in contrast with the Euclidean situation. In particular, a parametrization by curvature lines for the surfaces we are dealing with is not available in general.

For the class of immersions with constant f, the system (3.2) can be completely integrated.

Specifically, let $\psi : \Sigma \to \mathbb{L}^4$ be a timelike flat surface in the conditions of Theorem 3.1, and assume that $f = -2i\vartheta_1/\vartheta_2$ is constant. By effectuating the change (3.18) if necessary we may assume that $f = ai$ for $a \in \mathbb{R}$. Let $z : \Sigma \to \mathbb{C}$ be a global conformal parameter on Σ, so that Σ is identified with $\Omega = z(\Sigma) \subseteq \mathbb{C}$, and consider $h : \Omega \to \mathbb{C}$ the holomorphic function with $\vartheta_2 = 2h\,dz$. If we choose the holomorphic coordinate immersion ζ given by (3.25), then Eq. (3.22) shows that $g_{\zeta_1\zeta_2}=0$, where $\zeta=\zeta_1+i\zeta_2$. Thus the function $\phi(\zeta_1,\zeta_2)$ given by $\phi=g_{\bar\zeta}$ is of the form

$$\phi(\zeta_1,\zeta_2)=c(\zeta_1)+id(\zeta_2), \tag{4.3}$$

where $c(u),d(u)$ are arbitrary smooth real functions. Noting that

$$\zeta(z)=2\int h(z)\,dz, \tag{4.4}$$

we find from (4.3) that the general solution $\phi(z)$ to the differential system (3.2) is

$$\phi(z)=c\big(\zeta_1(z)\big)+id\big(\zeta_2(z)\big), \tag{4.5}$$

where $\zeta(z)$ is given by (4.4).

From (4.5) and $f=ai$, the regularity condition given by (3.17) turns into

$$\begin{cases} c''(\zeta_1(z))+2ac(\zeta_1(z))\neq 0,\\ d''(\zeta_2(z))-2ad(\zeta_2(z))\neq 0. \end{cases}$$

Consequently, if $c(u),d(u):\mathbb{R}\to\mathbb{R}$ satisfy $c''(u)+2ac(u)\neq 0$, $d''(u)-2ad(u)\neq 0$ for all $u\in\mathbb{R}$, the surface obtained from Theorem 3.1 and the data (f,h,ϕ) is regular.

Furthermore, from $f=ai$ the differential equation (3.23) is easily solved, and its solution (unique up to left multiplication by a constant element of $\mathbf{SL}(2,\mathbb{C})$) is the holomorphic curve in $\mathbf{SL}(2,\mathbb{C})$

$$F(\zeta)=\begin{pmatrix} \frac{i}{k\sqrt{2a}}\exp\big(i\sqrt{a/2}\zeta\big) & \frac{-1}{2k}\exp\big(i\sqrt{a/2}\zeta\big)\\ k\exp\big(-i\sqrt{a/2}\zeta\big) & -ik\sqrt{a/2}\exp\big(-i\sqrt{a/2}\zeta\big) \end{pmatrix}$$

for $k\in\mathbb{C}\setminus\{0\}$. That is, the solution to (3.1) is

$$F(z)=\begin{pmatrix} \frac{i}{k\sqrt{2a}}\exp\big(i\sqrt{a/2}\zeta(z)\big) & \frac{-1}{2k}\exp\big(i\sqrt{a/2}\zeta(z)\big)\\ k\exp\big(-i\sqrt{a/2}\zeta(z)\big) & -ik\sqrt{a/2}\exp\big(-i\sqrt{a/2}\zeta(z)\big) \end{pmatrix}, \tag{4.6}$$

where $\zeta(z)$ is given by (4.4). Therefore, all the timelike flat surfaces we are considering in this section are obtained in explicit coordinates by means of Theorem 3.1.

To study the completeness of these examples, we recall that the null coordinates of the surface are defined by means of (3.15) and (3.16), and that completeness holds if and only if (x,y) are globally defined on $\mathbb{L}^2$.

From (4.5) and $f=ai$, (3.15) and (3.16) are easily integrated to obtain

$$\begin{cases} x(z)+ay(z)=-\big(2a\int c+c'\big)(\zeta_1(z)),\\ x(z)-ay(z)=\big(2a\int d-d'\big)(\zeta_2(z)). \end{cases} \tag{4.7}$$

Here $\int c$, $\int d$ denote primitives of c,d. Thus, if $(x(z),y(z)):\Omega\to\mathbb{L}^2$ is surjective, the null coordinates are globally defined on $\mathbb{L}^2$ and the immersion is complete.

It is interesting to observe that, even if all data are defined over the whole plane $\mathbb{C}$, the surface might not be complete. For instance, the choices $\zeta(z)=z$, $a^2\neq 1/4$, $c(x)=d(x)=e^x$ produce a regular timelike flat surface $\psi:\mathbb{C}\to\mathbb{L}^4$ that is not complete, since $x+ay<0$, or alternatively $x+ay>0$, at every point.

However, it is easy to produce complete examples. Let $G_1(u),G_2(u):\mathbb{R}\to\mathbb{R}$ be two regular diffeomorphisms of $\mathbb{R}$, and let $c(u),d(u):\mathbb{R}\to\mathbb{R}$ be global solutions of the differential equations

$$c''(u)+2ac(u)=G_1'(u),\qquad d''(u)-2ad(u)=G_2'(u). \tag{4.8}$$

Since both equations are linear, such global solutions exist for any choice of initial data. Moreover, both differential equations can be explicitly integrated by means of the constants variation formula.

Then, if we assume that $h(z)$ is an entire function, we get that the data (f,h,ϕ) are defined over the whole complex plane $\mathbb{C}$. By Picard's great theorem $\zeta(z)$ omits at most one value in $\mathbb{C}$. Therefore, the conditions imposed to $G_1(u),G_2(u)$ ensure that the surface is regular, and that the coordinates (x,y) given by (4.7) are globally defined on $\mathbb{L}^2$ if and only if $\zeta(z)$ is surjective.

The converse of this construction also holds, as we verify next.

Theorem 4.1. *Let $E(z)$ be an entire function, $a \in \mathbb{R}$, choose $G_1(u), G_2(u) : \mathbb{R} \to \mathbb{R}$ two regular global diffeomorphisms of $\mathbb{R}$, and let $c(u), d(u) : \mathbb{R} \to \mathbb{R}$ be solutions of the linear differential equations (4.8), obtained by means of the constants variation formula. If the entire function*

$$\zeta(z) = \zeta_1(z) + i\zeta_2(z) = 2\int e^{E(z)}\,dz, \tag{4.9}$$

is surjective, the map $\psi : \mathbb{C} \to \mathrm{Herm}(2) \equiv \mathbb{L}^4$ given by $\psi = F\Lambda F^*$ where $\Lambda : \mathbb{C} \to \mathrm{Herm}(2)$ is

$$\Lambda = \begin{pmatrix} -\frac{1}{2}(c'(\zeta_1(z)) + d'(\zeta_2(z))) & c(\zeta_1(z)) - i d(\zeta_2(z)) \\ c(\zeta_1(z)) + i d(\zeta_2(z)) & a^{-1}(c'(\zeta_1(z)) - d'(\zeta_2(z))) \end{pmatrix}$$

and $F : \mathbb{C} \to \mathbf{SL}(2, \mathbb{C})$ is written as (4.6) by means of (4.9), is a (complete) isometric immersion of $\mathbb{L}^2$ into $\mathbb{L}^4$ with flat normal bundle, regular Gauss map, and whose holonomic frame is parallel.

Conversely, any (complete) isometric immersion of $\mathbb{L}^2$ into $\mathbb{L}^4$ with flat normal bundle, regular Gauss map and parallel holonomic frame is recovered in this way.

Proof. The direct part follows from Theorem 3.1 and the computations in this Section. Conversely, if $\psi : \Omega \subseteq \mathbb{C} \to \mathbb{L}^4$ is a complete timelike flat surface in the stated conditions, the function f is constant and can be chosen to be $f = ai$, $a \in \mathbb{R}$. In addition, completeness assures that the map $(x(z), y(z)) : \Omega \subseteq \mathbb{C} \to \mathbb{L}^2$ defined by means of (4.7) is surjective. This indicates that $\zeta(z)$ given by (3.25) is surjective, and thus we must have $\Omega = \mathbb{C}$, i.e. the conformal structure of the surface is parabolic. In addition, as h is a non-vanishing function, it is of the form $h(z) = e^{E(z)}$ for $E(z)$ an entire function. Then $\zeta(z)$ writes down as in (4.9), and is a surjective entire function. In particular, since the function ϕ from Theorem 3.1 is defined in the whole plane $\mathbb{C}$, and is given by (4.5), the real functions $c(u), d(u)$ appearing in that expression must be globally defined on $\mathbb{R}$. Once here, the regularity condition and the completeness of the surface (described in (4.7)) assure that the functions $G_1(u), G_2(u)$ given by (4.8) must be regular global diffeomorphisms of $\mathbb{R}$. This completes the proof. □

5. Flat timelike surfaces in de Sitter 3-space

Let $\psi : \Sigma \to \mathbb{S}^3_1$ be a flat timelike surface with unit normal in $\mathbb{S}^3_1$ given by $\eta : \Sigma \to \mathbb{H}^3$. Then ψ has regular Gauss map (since its scalar second fundamental form is a definite $(2,0)$-tensor) and flat normal bundle, and we can choose $\eta = N$ and $\psi = \hat{N}$ as the parallel orthonormal frame of the normal bundle of ψ in $\mathbb{L}^4$. With this, the Weierstrass representation in Theorem 3.1 yields a conformal representation for flat timelike surfaces in $\mathbb{S}^3_1$:

Corollary 5.1. *Let Σ be a Riemann surface, and consider two nowhere vanishing holomorphic 1-forms ϑ_1, ϑ_2 on Σ verifying $\mathrm{Im}(\vartheta_1/\vartheta_2) \neq 0$. If $F : \Sigma \to \mathbf{SL}(2, \mathbb{C})$ is a holomorphic curve satisfying*

$$F^{-1}\,dF = \begin{pmatrix} 0 & \vartheta_1 \\ \vartheta_2 & 0 \end{pmatrix} \tag{5.1}$$

on Σ, then the map

$$\psi = F \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} F^* : \Sigma \to \mathrm{Herm}(2) \equiv \mathbb{L}^4 \tag{5.2}$$

is a timelike flat surface in $\mathbb{S}^3_1$ for which $\mathcal{G}^+, \mathcal{G}^- : \Sigma \to \mathbb{C} \cup \{\infty\}$ are $\mathcal{G}^+ = F_{11}/F_{21}$, $\mathcal{G}^- = F_{12}/F_{22}$.

Conversely, every simply connected timelike flat surface in $\mathbb{S}^3_1$ can be represented in this way.

Proof. It follows from Theorem 3.1, taking into account that the condition $\psi = \hat{N}$ provides $\phi = i$, and that the regularity condition (3.17) turns into $\mathrm{Re}\, f \neq 0$, which we can also write down as $\mathrm{Im}(\vartheta_1/\vartheta_2) \neq 0$. □

As a consequence of this conformal representation, we give an alternative proof of the following classical result by Dajczer and Nomizu:

Corollary 5.2. [6] *The only isometric immersions of $\mathbb{L}^2$ into $\mathbb{S}^3_1$ are hyperbolic cylinders.*

Proof. Let $\psi:\mathbb{L}^2\to\mathbb{S}^3_1$ be an isometric immersion, and recall the notations in Theorem 3.1. As we have said, ψ is a parallel normal section, and we may assume that $\psi=\hat{N}$. From the structure equations (2.4) and the Gauss equation in (2.5) we get $\hat{\varphi}_{xx}=\hat{\varphi}_{yy}=0$, $\hat{\varphi}_{xy}=-1$, and $\varphi_{xx}\varphi_{yy}-\varphi^2_{xy}=1$. In addition, recalling the complex parameters ζ, w introduced in Theorem 2.3, we see that

$$z=\zeta+w=(-x+\varphi_x)+i(-y+\varphi_y) \tag{5.3}$$

is a global holomorphic coordinate immersion. Since $\varphi_{xx}\varphi_{yy}-\varphi^2_{xy}=1$, Lewy's Lemma (see [17]) shows that $(x,y)\mapsto z(x,y)$ increases distances. In particular $z:\mathbb{L}^2\to\mathbb{C}$ is a global diffeomorphism, and then z is a proper conformal coordinate and the surface is conformally equivalent to $\mathbb{C}$.

Finally, as $\operatorname{Re} f\neq 0$, and f is an entire function, it must be constant. Besides, since $dz=d\zeta+dw$ and $d\zeta=2h\,dz$, the function h is also constant, $h=(2+2f)^{-1}$ (observe that since z is a diffeomorphism, $f\neq -1$). Hence, the surface ψ is recovered as (5.2), where $F:\mathbb{C}\to\mathbf{SL}(2,\mathbb{C})$ is given by (4.6) for $\zeta(z)=bz$, $b\in\mathbb{C}\setminus\{0\}$. But once here, the coordinates of ψ satisfy the relations $-\psi_0^2+\psi_3^2=\text{const.}$ and $\psi_1^2+\psi_2^2=\text{const}$, i.e. the surface is a hyperbolic cylinder. $\square$

On the other hand, it is well known that totally umbilical spacelike surfaces in $\mathbb{S}^3_1$ with vanishing Gauss curvature are given by

$$M^k_v=\{x\in\mathbb{S}^3_1:\ \langle x,v\rangle=k\},$$

where $v\in\mathbb{L}^4$ is a null vector and k is a non-zero real number.

Any Lorentzian hyperbolic cylinder satisfies that there exists a fixed null vector v such that the foliation $\mathcal{F}=\{M^k_v:\ k\in\mathbb{R}^*\}$ intersects each point of the hyperbolic cylinder at a constant angle. More generally, we prove that this fact characterizes hyperbolic cylinders as follows

Theorem 5.3. *Let $\psi:\Sigma\longrightarrow\mathbb{S}^3_1$ be a Lorentzian flat immersion of a compact surface with boundary $\partial\Sigma$ (not necessarily connected) into $\mathbb{S}^3_1$, and let v be a null vector satisfying that each point of $\psi(\Sigma)$ intersects the foliation $\mathcal{F}=\{M^k_v:\ k\in\mathbb{R}^*\}$. If $\mathcal{F}$ cuts $\psi(\partial\Sigma)$ at a constant angle, then $\psi(\Sigma)$ is a piece of a hyperbolic cylinder.*

This theorem has been motivated by a result in [16] for flat surfaces in the hyperbolic 3-space.

Proof. We can assume that, up to an isometry of $\mathbb{S}^3_1$, $v=(-1,0,0,1)$. Thus, our foliation is made up of the totally umbilical surfaces given by

$$M^k_v=\{(x_0,x_1,x_2,x_3)\in\mathbb{S}^3_1:\ x_0+x_3=k\},\quad k\in\mathbb{R}^*.$$

Now, let us denote by $\eta=(\eta_0,\eta_1,\eta_2,\eta_3)$ the unit normal of the immersion ψ. Then, since the unit normal of M^k_v at a point $p=(x_0,x_1,x_2,x_3)$ is

$$N_p=\left(x_0+\frac{1}{x_0+x_3},x_1,x_2,x_3-\frac{1}{x_0+x_3}\right),$$

the inner product of η and N at $p\in\psi(\Sigma)$ is given by

$$\langle\eta,N\rangle=-\frac{\eta_0+\eta_3}{x_0+x_3}.$$

Hence, from (3.9) and (5.2), if we consider the holomorphic curve

$$F=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathbf{SL}(2,\mathbb{C}),$$

we see that

$$\frac{\eta_0+\eta_3}{x_0+x_3}=\frac{i\overline{A}B-iA\overline{B}}{A\overline{B}+\overline{A}B}=\frac{\operatorname{Re}(A\overline{B})}{\operatorname{Im}(A\overline{B})}$$

on Σ. Moreover, given a local parameter z of the Riemann surface Σ, we obtain

$$\frac{\partial^2}{\partial z\partial \bar{z}}\left(\arctan\left(\frac{i\overline{A}B - iA\overline{B}}{A\overline{B} + \overline{A}B}\right)\right) = 0.$$

In this way, the function $\arctan((\eta_0 + \eta_3)/(x_0 + x_3))$ is harmonic on Σ and constant along $\partial\Sigma$. Therefore, $\operatorname{Re}(A\overline{B})/\operatorname{Im}(A\overline{B})$ is constant on Σ and using that A and B are holomorphic we obtain that there exists a constant c_0 such that $B = c_0 A$.

Taking a new local parameter ζ, if necessary, we can assume that $h(\zeta) = 1/2$. Thus, from (3.20), we have $B = A_\zeta$, that is, $A(\zeta) = c_1 e^{c_0\zeta}$ with $c_1 \in \mathbb{C}^*$. At last, using (3.19) we obtain $f(\zeta) = -2ic_0^2$. Therefore, f is constant on Σ, or equivalently, $\psi(\Sigma)$ is a piece of a hyperbolic cylinder. $\square$

Acknowledgements

The first author is partially supported by Junta de Comunidades de Castilla-La Mancha, Grant No PAI-05-034. The authors are partially supported by MCYT-FEDER, Grant No MTM2004-02746.

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